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THICKNESS-SHEAR AND FLEXURAL VIBRATIONS OF PARTIALLY PLATED, CRYSTAL PLATES

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Abstract—In this paper, mathematical solutions are exhibited which predict the influence of length and mass, of rectangular electrode platings, on the sharpness of the fundamental thickness-shear resonance and on the number and distribution of anharmonic overtones between the cut-off frequencies of the plated and unplated portions of AT-type quartz plates.

1. INTRODUCTION

TWO INTERESTING and useful effects of the size of electrode platings on vibrations of AT-type quartz plates have been employed, principally by Bechmann [1], to improve the performance of the crystals in filter circuits. First: if electrode platings, symmetrically disposed on the two faces of an AT-type quartz plate, cover only a central portion of the plate and if the plate is excited in a thickness-shear resonance, much of the motion of the plate is confined to the region under the electrodes. Second: as the area of the electrodes is reduced, the responses of the anharmonic overtones of thickness-shear are diminished, successively, until, below a certain length/thickness ratio (Bechmann's Number), the response of the fundamental is, essentially, all that remains.

The first of these phenomena was given a qualitative, physical explanation, by Shockley, Curran and Koneval [2], on the basis of the dispersion relations for elastic waves in quartz plates [3, 4, 5]. Consider the dispersion curve (TS_1 in Fig. 1) for a straight-crested, thickness-shear wave progressing in the direction of the axis of digonal symmetry in the plane of an AT-type quartz plate without electrodes. At zero wave-number there is a cut-off circular frequency, ω_1, below which the wave-number is imaginary; i.e. at frequencies below ω_1, the wave is non-propagating and decays exponentially with distance. Alternatively, consider the analogous curve for a plate coated with electrodes. Owing to the mass loading of the electrodes, the frequency is lowered throughout (curve $\overline{TS}_1$ in Fig. 1). In particular, the cut-off frequency $\bar{\omega}_1$ is less than ω_1. Now suppose that only a central portion of the plate is coated with electrodes across which the impressed voltage has a frequency between ω_1 and $\bar{\omega}_1$. A propagating thickness-shear wave is excited in the part of the plate under the electrodes; but the corresponding wave at the same frequency, in the remainder of the plate, decays exponentially: resulting in a total reflection phenomenon whereby thickness-shear vibrational energy is trapped in the region under the electrodes.

Now, flexural, face-shear and extensional waves can propagate at any frequency. These waves are excited, at the boundary between the plated and unplated portions of the plate, and propagate into both regions; so that, even at frequencies between $\bar{\omega}_1$ and ω_1, some of the energy escapes from the plated region—mostly in the outgoing flexural

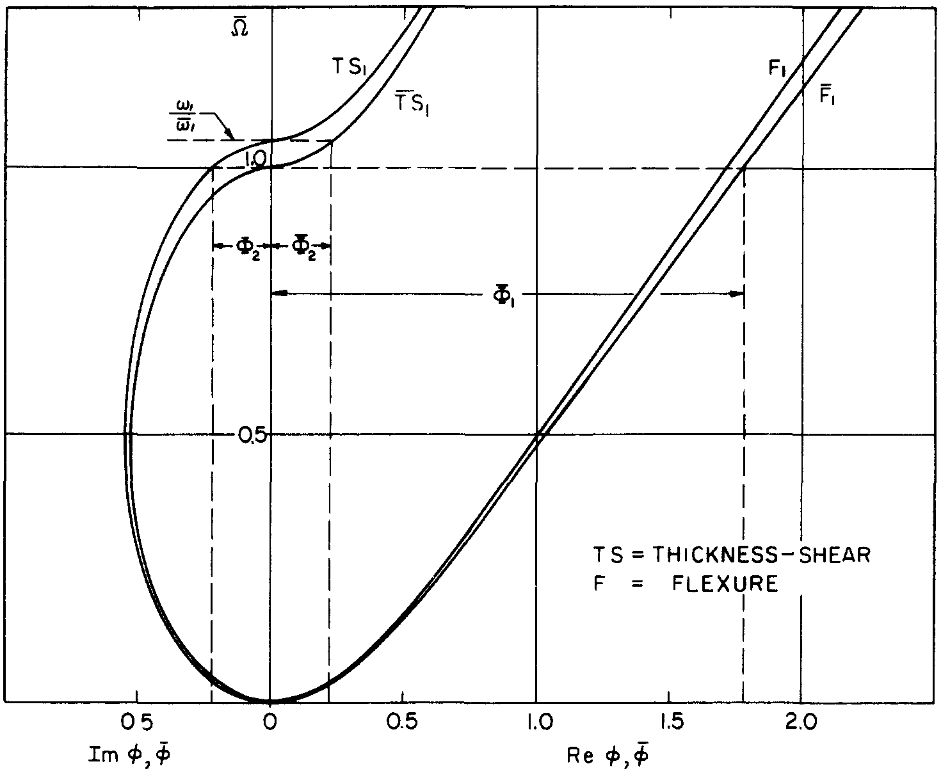

FIG. 1. Dispersion curves for straight-crested, thickness-shear and flexural waves with wave-normals in the direction of the digonal axis in unplated (TS_1 and F_1) and plated ($\overline{TS}_1$ and $\bar{F}_1$) AT-type quartz plates.

wave if the edge of the electrode is straight and perpendicular to the digonal axis in the plane of the plate. It is shown, in this paper, how the energy in the escaping flexural wave depends on the dimensions and material properties of the electrodes and the crystal in the case of rectangular electrodes. The Q of the resonance (proportional to the ratio of the energy trapped to the the energy lost) is found to be an almost periodic function of the ratio of the length of the electrode, in the digonal direction, to the thickness of the plate. The Q is a maximum when the length of the electrode is approximately an integral multiple of the wavelength of the flexural wave in the plated portion of the plate; i.e. there is little transmission of flexural energy across the boundary between the plated and unplated portions when there is a node of flexure there. The minima of Q become lower as the length of the electrodes is reduced and as the thickness of the electrodes is increased.

An explanation of the phenomenon of Bechmann's Number can be based on the observations that only the deformation in the portion of the plate under the electrodes contributes to the current through the crystal and only those anharmonic overtones with frequencies less than ω_1 are trapped under the electrodes. The anharmonic overtone modes of thickness-shear, in the plated portion of the plate, have successively shorter and shorter half-wave-lengths: approximately equal to $\frac{1}{3}, \frac{1}{5}, \frac{1}{7} \ldots$ of the length of a rectangular electrode in the digonal direction. To each successively higher anharmonic overtone mode, there corresponds an only slightly higher frequency; so that, for a long enough electrode, many anharmonic overtone modes can have frequencies less than ω_1 and, hence, can be totally trapped. A forcing frequency, equal to that of a trapped overtone mode, will produce a relatively large amplitude of thickness-shear deformation in the plated portion of the plate because much of the input energy is trapped there; but a forcing frequency equal to that of an anharmonic overtone resonance above ω_1 excites the whole plate because only a small proportion of that mode is reflected at the edge of the electrode. Consequently, a smaller proportion of the total energy is to be found in the part of the plate under the electrodes—the part which contributes to the current through the crystal. Now, if the electrode is small enough, the wave-length of the first anharmonic overtone mode will be so short that the frequency of that mode, and all the higher ones, will be greater than ω_1 and there will be no total trapping of any anharmonic overtone modes—only the fundamental can be trapped. Thus, the critical length of electrode is simply the wave-length of a thickness-shear wave, in the plated portion of the plate, at the cut-off frequency of the thickness-shear wave in the unplated portion of the plate. The ratio of this critical wave-length to the thickness of the plate is Bechmann's Number.

Curran and Koneval [6] have made a series of experiments with rectangular electrodes of successively smaller lengths. Their data of number and frequency of trapped anharmonic overtones *versus* length of electrode are compared, in the second part of this paper, with the results of a mathematical solution. Their formula for Bechmann's Number, in terms of $\bar{\omega}_1/\omega_1$ and a numerical constant, is compared with a formula, obtained mathematically, expressed in terms of the dimensions and material properties of the electrodes and the crystal.

2. THICKNESS-SHEAR AND FLEXURAL MOTIONS OF INFINITE, UNPLATED AND PLATED PLATES

It has been shown [3, 4] that the motions of crystal plates are well described, up to a frequency somewhat higher than the cut-off frequency of the lowest thickness-shear mode, by solutions of a system of five, coupled, two-dimensional, second order partial differential equations governing three components of displacement and two components of rotation. Quartz is a crystal of the trigonal trapezohedral class [7]: with one trigonal axis and, in the plane at right angles, three digonal axes. A rotated Y-cut plate [7] contains a digonal axis and is cut at an angle (35° 15′ for the AT-cut) to the trigonal axis. In the case of straight-crested waves advancing in the direction of the digonal axis in the plane of a rotated Y-cut plate, three equations of coupled thickness-shear, flexure and face-shear separate from the five coupled equations [3]. If, in these three, the coupling with face-shear is neglected, the remaining equations of coupled thickness-shear and flexure [8] have the same form as Timoshenko's equations of flexural vibrations of beams in which transverse shear deformation and rotatory inertia are taken into account [9]:

$$k_1^2 c_{66}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial \psi_x}{\partial x}\right) = \rho \frac{\partial^2 v}{\partial t^2}, \tag{1a}$$

$$\frac{\gamma_{11} h^2}{3}\frac{\partial^2 \psi_x}{\partial x^2} - k_1^2 c_{66}\left(\frac{\partial v}{\partial x} + \psi_x\right) = \frac{\rho h^2}{3}\frac{\partial^2 \psi_x}{\partial t^2}, \tag{1b}$$

where x is the digonal axis in the plane of the plate, v is the deflection of the plate, ψ_x is the rotation of a line element initially normal to the middle plane of the plate, ρ is the density, $2h$ is the thickness of the plate, k_1 is the shear-correction factor, c_{66} is the shear modulus of elasticity in the plane normal to the plate through the x-axis and γ_{11} is Voigt's stretch modulus of the plate in the x-direction. In terms of the compliances, s_{pq}, and stiffnesses, c_{pq}, referred to rectangular coordinates x, y, z, with y normal to the middle plane of the plate,

$$\begin{aligned}\gamma_{11} &= s_{33}/(s_{11}s_{33} - s_{13}^2),\\ &= c_{11} - c_{12}^2/c_{22} - (c_{14} - c_{12}c_{24}/c_{22})^2/(c_{44} - c_{24}^2/c_{22}).\end{aligned}$$

For the AT-cut of quartz,

$$c_{66} = 29{\cdot}01, \qquad \gamma_{11} = 85{\cdot}93$$

in units of 10^{10} dyne/cm^2, as calculated from Bechmann's values of the principal constants [10].

As in Timoshenko's equations, the transverse shear force, V_x, and the bending moment, M_x, both per unit width of plate, are given by

$$V_x = 2hk_1^2c_{66}\left(\frac{\partial v}{\partial x} + \psi_x\right), \qquad M_x = \frac{2h^3\gamma_{11}}{3}\frac{\partial \psi_x}{\partial x}. \tag{2}$$

Consider, now, the same plate with electrode platings, on each face, of like thicknesses $2h'$ and density ρ'. An alternating voltage, impressed across the electrodes, is mechanically equivalent to a couple, C per unit area, distributed uniformly over the plate [11, 12]. If the remaining coupling with the electric field is neglected and if the stiffnesses of the platings are neglected, the equations analogous to (1) are [13]

$$\bar{k}_1^2c_{66}\left(\frac{\partial^2\bar{v}}{\partial x^2} - \frac{\partial\bar{\psi}_x}{\partial x}\right) = \rho(1 + R)\frac{\partial^2\bar{v}}{\partial t^2}, \tag{3a}$$

$$\frac{\gamma_{11}h^2}{3}\frac{\partial^2\bar{\psi}_x}{\partial x^2} - \bar{k}_1^2c_{66}\left(\frac{\partial\bar{v}}{\partial x} + \bar{\psi}_x\right) + \frac{C}{2h} = \frac{\rho h^2(1 + 3R)}{3}\frac{\partial^2\bar{\psi}_x}{\partial t^2}, \tag{3b}$$

where barred symbols pertain to a plated plate and

$$R = 2\rho'h'/\rho h, \tag{4}$$

i.e. R is the ratio of the mass per unit area of both electrodes to the mass per unit area of the plate. The transverse shear force and bending moment, per unit width of plated plate, are

$$\bar{V}_x = 2h\bar{k}_1^2c_{66}\left(\frac{\partial\bar{v}}{\partial x} + \bar{\psi}_x\right), \qquad \bar{M}_x = \frac{2h^3\gamma_{11}}{3}\frac{\partial\bar{\psi}_x}{\partial x}. \tag{5}$$

The shear correction factor $\bar{k}_1$ is expressed in terms of R by equating the thickness-shear cut-off frequency, obtained from (3), with that obtained from a solution of the three-dimensional equations of elasticity. In the case of (3), we set

$$C = \bar{v} = 0, \qquad \bar{\psi}_x = \bar{A}\exp(i\bar{\omega}_1 t),$$

where $\bar{A}$ is a constant. Then

$$\bar{\omega}_1^2 = 3\bar{k}_1^2 c_{66}/\rho h^2(1 + 3R). \tag{6}$$

For simple thickness-shear motion, the three-dimensional equations reduce to

$$c_{66}\frac{\partial^2 u}{\partial y^2} = \rho\frac{\partial^2 u}{\partial t^2}, \qquad T_{xy} = c_{66}\frac{\partial u}{\partial y}, \tag{7}$$

where y is the coordinate normal to the middle plane of the plate, u is the displacement in the direction of x and T_{xy} is the shear stress on planes normal to x or y. For

$$u = A \sin \eta y \exp(i\omega t), \tag{8}$$

the first of (7) yields

$$\omega^2 = \eta^2 c_{66}/\rho. \tag{9}$$

As boundary conditions we take

$$T_{xy} = \mp 2\rho' h' \frac{\partial^2 u}{\partial t^2} \quad \text{on} \quad y = \pm h, \tag{10}$$

i.e. the inertia of the electrode plating is balanced by a shear traction on the surface of the plate. With (8) and the second of (7), (10) becomes

$$c_{66}\eta \cos \eta h = 2h'\rho'\omega^2 \sin \eta h;$$

or, with (4) and (9),

$$R\eta h \tan \eta h = 1. \tag{11}$$

For $R \ll 1$, the smallest root of (11) is given, approximately, by

$$(\eta h)_1 = \pi/2(1 + R). \tag{12}$$

Upon substituting (12) into (9) and equating the resulting expression for ω^2 to (6), we find

$$\bar{k}_1^2 = \pi^2(1 + 3R)/12(1 + R)^2, \qquad \bar{\omega}_1^2 = \pi^2 c_{66}/4\rho h^2(1 + R)^2. \tag{13}$$

The corresponding shear correction factor and cut-off frequency for the unplated plate ($R = 0$) are

$$k_1^2 = \pi^2/12, \qquad \omega_1^2 = \pi^2 c_{66}/4\rho h^2. \tag{14}$$

Thus, the effect of the mass of the electrodes is equivalent to increases of transverse inertia, rotatory inertia and shear stiffness by factors $1 + R$, $1 + 3R$ and $(1 + 3R)/(1 + R)^2$, respectively.

From (13) and (14),

$$\omega_1/\bar{\omega}_1 = 1 + R. \tag{15}$$

As shown in [14], inclusion of piezoelectric properties would make $\omega_1/\bar{\omega}_1$ slightly greater than $1 + R$; but this effect is omitted here.

Two fundamental solutions of the equations of motion, (1) and (3), are recorded here for later use. First: for an applied voltage mechanically equivalent to $C_0 \cos \omega t$, with C_0 a constant, a particular solution of (3) is $\bar{v} = 0$ and

$$\bar{\psi}_x = \bar{\psi}_0 \cos \omega t, \qquad \bar{\psi}_0 = C_0/2h\bar{k}_1^2 c_{66}(1 - \omega^2/\bar{\omega}_1^2). \tag{16}$$

Second: (1) and (3) admit the free wave solutions

$$\psi_x = A \exp[i(\xi x - \omega t)], \qquad v = i\alpha A h \exp[i(\xi x - \omega t)],$$
$$\bar{\psi}_x = \bar{A} \exp[i(\bar{\xi} x - \omega t)], \qquad \bar{v} = i\bar{\alpha}\bar{A} h \exp[i(\bar{\xi} x - \omega t)],$$

if the amplitude ratios, α and $\bar{\alpha}$, satisfy the equations

$$\alpha = \frac{\xi h}{\xi^2 h^2 - 3\Omega^2} = \frac{1 + \hat{\gamma}_{11}\xi^2 h^2 - \Omega^2}{\xi h}, \tag{17}$$

$$\bar{\alpha} = \frac{\bar{\xi} h}{\bar{\xi}^2 h^2 - 3r\bar{\Omega}^2} = \frac{1 + \tilde{\gamma}_{11}\bar{\xi}^2 h^2 - \bar{\Omega}^2}{\bar{\xi} h}, \tag{18}$$

where

$$\Omega = \omega/\omega_1, \qquad \hat{\gamma}_{11} = \gamma_{11}/3k_1^2 c_{66},$$
$$\bar{\Omega} = \omega/\bar{\omega}_1, \qquad \tilde{\gamma}_{11} = \gamma_{11}/3\bar{k}_1^2 c_{66},$$
$$r = (1 + R)/(1 + 3R).$$

The second of (17) and (18) yield the dispersion relations depicted in Fig. 1 with dimensionless frequency, $\bar{\Omega}$, as ordinate and dimensionless wave-number, $\varphi = \xi h$ or $\bar{\varphi} = \bar{\xi} h$, as abscissa. Points on the curves marked TS_1 and $\overline{TS_1}$ correspond to motions in the unplated and plated plates, respectively, which are predominantly thickness-shear ($|\alpha| < 1$, $|\bar{\alpha}| < 1$). The curves marked F_1 and $\bar{F}_1$ give the dispersion relations for predominantly flexural motion ($|\alpha| > 1$, $|\bar{\alpha}| > 1$). It will be observed that, in the frequency range $\bar{\omega}_1 < \omega < \omega_1$, i.e. $1 < \bar{\Omega} < \omega_1/\bar{\omega}_1$, the thickness-shear waves are propagating waves ($\bar{\varphi}$ real) in the plated plate but non-propagating waves (φ imaginary) in the unplated plate; whereas flexural waves in both plates are propagating waves at all frequencies.

3. EFFECT OF SIZE OF ELECTRODES ON Q

Suppose that the strip $-a \leqslant x \leqslant a$, on each face of an infinite plate, is coated with electrodes, Fig. 2, across which is impressed a voltage mechanically equivalent to $C_0 \cos \omega t$ with $\bar{\omega}_1 \leqslant \omega \leqslant \omega_1$. A suitable form of solution of (1), for the unplated portion $x \geqslant a$, is

$$\psi_x = A_1 \sin(\xi_1 x - \xi_1 a - \omega t) + A_2 \exp[-\xi_2(x - a)] \cos \omega t$$
$$+ B_1 \cos(\xi_1 x - \xi_1 a - \omega t) + B_2 \exp[-\xi_2(x - a)] \sin \omega t, \tag{19a}$$

$$v = -\alpha_1 A_1 h \cos(\xi_1 x - \xi_1 a - \omega t) + \alpha_2 A_2 h \exp[-\xi_2(x - a)] \cos \omega t$$
$$+ \alpha_1 B_1 h \sin(\xi_1 x - \xi_1 a - \omega t) + \alpha_2 B_2 h \exp[-\xi_2(x - a)] \sin \omega t. \tag{19b}$$

For the unplated portion $x \leqslant -a$, x and v, in (19), are changed to $-x$ and $-v$. For the plated portion $-a \leqslant x \leqslant a$, we take

$$\bar{\psi}_x = \bar{\psi}_0 \cos \omega t + (\bar{A}_1 \cos \bar{\xi}_1 x + \bar{A}_2 \cos \bar{\xi}_2 x) \cos \omega t$$
$$+ (\bar{B}_1 \cos \bar{\xi}_1 x + \bar{B}_2 \cos \bar{\xi}_2 x) \sin \omega t, \tag{20a}$$

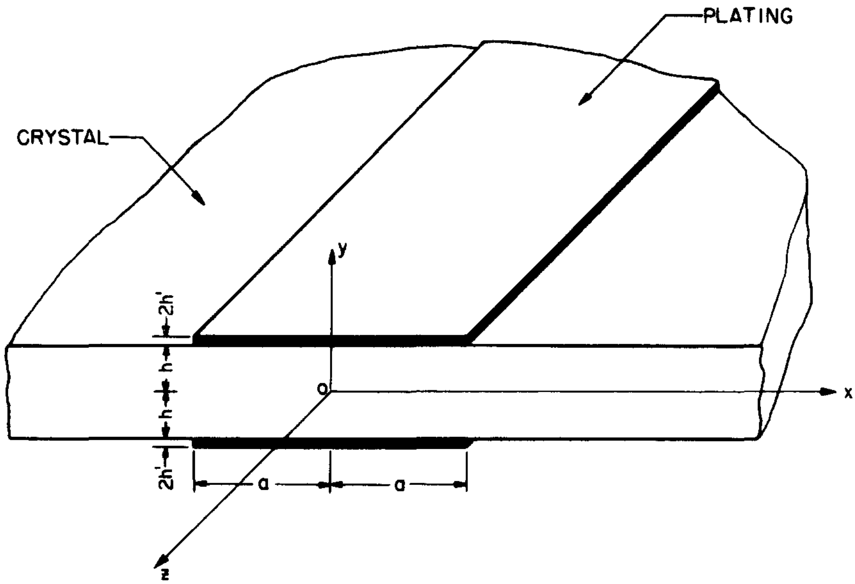

FIG. 2. Partially plated, infinite plate.

$$\bar{v} = (\bar{\alpha}_1 \bar{A}_1 h \sin \bar{\xi}_1 x + \bar{\alpha}_2 \bar{A}_2 h \sin \bar{\xi}_2 x) \cos \omega t + (\bar{\alpha}_1 \bar{B}_1 h \sin \bar{\xi}_1 x + \bar{\alpha}_2 \bar{B}_2 h \sin \bar{\xi}_2 x) \sin \omega t, \tag{20b}$$

where $\bar{\psi}_0$ is given by (16).

The functions (19) and (20) are solutions of (1) and (3), respectively, if

$$\alpha_i = \frac{\varphi_i}{3\Omega^2 + (-1)^i \varphi_i^2} = \frac{(-1)^i(1 - \Omega^2) - \hat{\gamma}_{11}\varphi_i^2}{\varphi_i}, \qquad i = 1, 2, \tag{21}$$

$$\bar{\alpha}_i = \frac{\bar{\varphi}_i}{3r\bar{\Omega}^2 - \bar{\varphi}_i^2} = \frac{\bar{\Omega}^2 - 1 - \bar{\gamma}_{11}\bar{\varphi}_i^2}{\bar{\varphi}_i}, \qquad i = 1, 2, \tag{22}$$

where

$$\varphi_i = \xi_i h, \qquad \bar{\varphi}_i = \bar{\xi}_i h.$$

The second of (21) and (22) give the dispersion relations in the form of biquadratic equations from which we select, as roots, the positive roots of

$$\varphi_i^2 = \tfrac{1}{2}\hat{\gamma}_{11}^{-1}\Omega^2\{[(1 + 3\hat{\gamma}_{11})^2 + 12\hat{\gamma}_{11}(\Omega^{-2} - 1)]^{\frac{1}{2}} - (-1)^i(1 + 3\hat{\gamma}_{11})\}, \tag{23}$$

$$\bar{\varphi}_i^2 = \tfrac{1}{2}\bar{\gamma}_{11}^{-1}\bar{\Omega}^2\{1 + 3r\bar{\gamma}_{11} - (-1)^i[(1 + 3r\bar{\gamma}_{11})^2 - 12r\bar{\gamma}_{11}(1 - \bar{\Omega}^{-2})]^{\frac{1}{2}}\}. \tag{24}$$

With these choices, ξ_1, ξ_2, $\bar{\xi}_1$ and $\bar{\xi}_2$ are all real and positive in the frequency range $\bar{\omega}_1 < \omega < \omega_1$, i.e. $\Omega < 1$ and $\bar{\Omega} > 1$. Hence, in that range, the solutions (19) and (20) have the following properties. In the unplated portion, there are two flexural waves (A_1, B_1) ninety degrees out of phase and propagating away from the electrodes; and two thickness-shear vibrations (A_2, B_2) ninety degrees out of phase and decaying away from the electrodes. In the plated portion, there are a forced thickness-shear vibration $\bar{\psi}_0 \cos \omega t$

and four additional vibrational motions: two of flexure ($\bar{A}_1, \bar{B}_1$) ninety degrees out of phase and two of thickness-shear ($\bar{A}_2, \bar{B}_2$) ninety degrees out of phase.

The conditions of continuity at the boundaries of the plated and unplated portions of the plate require that, on $x = \pm a$,

$$\psi_x = \bar{\psi}_x, \qquad v = \bar{v}, \qquad M_x = \bar{M}_x, \qquad V_x = \bar{V}_x.$$

These four conditions lead to eight equations on the eight constants $A_i, B_i, \bar{A}_i, \bar{B}_i$:

$$\begin{bmatrix} a_{11} & a_{12} & 1 & 0 & 1 & 0 & 0 & 0 \\ a_{21} & a_{22} & \beta_2 & 0 & \beta_1 & 0 & 0 & 0 \\ a_{31} & a_{32} & -\alpha_2 & \alpha_1 & 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & -\varphi_2 & \varphi_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & b_{11} & b_{12} \\ 0 & 0 & 0 & -\beta_1 & 0 & \beta_2 & b_{21} & b_{22} \\ 0 & 0 & 0 & 0 & \alpha_1 & -\alpha_2 & b_{31} & b_{32} \\ 0 & 0 & 0 & 0 & \varphi_1 & -\varphi_2 & b_{41} & b_{42} \end{bmatrix} \begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \\ A_2 \\ A_1 \\ B_1 \\ B_2 \\ \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} = \bar{\psi}_0 \begin{bmatrix} 1 \\ k \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{25}$$

where

$$\begin{aligned} a_{11} &= b_{11} = -\cos\bar{\varphi}_1\lambda & a_{31} &= b_{31} = \bar{\alpha}_1 \sin\bar{\varphi}_1\lambda \\ a_{12} &= b_{12} = -\cos\bar{\varphi}_2\lambda & a_{32} &= b_{32} = \bar{\alpha}_2 \sin\bar{\varphi}_2\lambda \\ a_{21} &= b_{21} = -k\bar{\beta}_1 \cos\bar{\varphi}_1\lambda & a_{41} &= b_{41} = \bar{\varphi}_1 \sin\bar{\varphi}_1\lambda \\ a_{22} &= b_{22} = -k\bar{\beta}_2 \cos\bar{\varphi}_2\lambda & a_{42} &= b_{42} = \bar{\varphi}_2 \sin\bar{\varphi}_2\lambda \\ \beta_i &= 1 - (-1)^i\alpha_i\varphi_i & \bar{\beta}_i &= 1 + \bar{\alpha}_i\bar{\varphi}_i \\ \lambda &= a/h & k &= \bar{k}_1^2/k_1^2 \end{aligned}$$

For given forcing frequency ω, material properties of plate and electrodes, mass ratio R and dimensional ratio λ, all of the coefficients in (25) are known and the amplitudes A_i, $\bar{A}_i$, B_i, $\bar{B}_i$ can be computed.

The quantity of present interest is the Q of the partially plated plate. This is defined as 2π times the ratio of the energy per cycle in the plated portion of the plate to the energy per cycle in the escaping flexural waves—at resonance. Hence, to compute Q, the frequency at resonance must be known. Now, resonance is identified by a maximum of the current through the crystal. The current is proportional to the integral of the surface charge over the electrodes; and the surface charge, in turn, is a linear function of the thickness-shear strain $\partial\bar{v}/\partial x + \bar{\psi}_x$ [12]. Hence the criterion for resonance is a maximum of the amplitude of

$$\int_{-a}^{a} (\partial\bar{v}/\partial x + \bar{\psi}_x)\,dx. \tag{26}$$

Upon substituting (20) into (26) and performing the integration, we find that the amplitude of (26) is proportional to

$$S = (\bar{\Omega}^2 - 1)^{-1}(S_c^2 + S_s^2)^{\frac{1}{2}} \tag{27}$$

where

$$S_c = \lambda + \bar{A}_1\bar{\beta}_1\bar{\varphi}_1^{-1}\sin\bar{\varphi}_1\lambda + \bar{A}_2\bar{\beta}_2\bar{\varphi}_2^{-1}\sin\bar{\varphi}_2\lambda,$$

$$S_s = \bar{B}_1\bar{\beta}_1\bar{\varphi}_1^{-1}\sin\bar{\varphi}_1\lambda + \bar{B}_2\bar{\beta}_2\bar{\varphi}_2^{-1}\sin\bar{\varphi}_2\lambda.$$

For given material properties and given R and λ, the amplitudes $A_i, B_i, \bar{A}_i, \bar{B}_i$ can be computed, from (25), for trial values of dimensionless frequency $\bar{\Omega}$; and the values of $\bar{\Omega}$ that give maxima of S can be identified. The process can then be repeated over a range of values of λ. The result is a spectrum of resonance frequencies *versus* λ. The range of trial frequencies can be narrowed by centering on the approximate values obtained by the simplified procedure described in Section 4.

Next, the energies must be computed. The kinetic and potential energies, per cycle, in the plated portion of the plate are given by [13]

$$K = \frac{1}{T}\int_0^T\int_{-a}^{a}\left[\rho h(1+R)\left(\frac{\partial\bar{v}}{\partial t}\right)^2 + \tfrac{1}{3}\rho h^3(1+3R)\left(\frac{\partial\bar{\psi}_x}{\partial t}\right)^2\right]\mathrm{d}x\,\mathrm{d}t,$$

$$U = \frac{1}{T}\int_0^T\int_{-a}^{a}\left[\bar{k}_1^2 c_{66} h\left(\frac{\partial\bar{v}}{\partial x} + \bar{\psi}_x\right)^2 + \tfrac{1}{3}\gamma_{11}h^3\left(\frac{\partial\bar{\psi}_x}{\partial x}\right)^2\right]\mathrm{d}x\,\mathrm{d}t,$$

where T $(= 2\pi/\omega)$ is the period. Upon performing the integrations, we find

$$\begin{aligned}(\tfrac{1}{2}h_1^2\bar{k}_1^2 c_{66}\bar{\Omega}^2)^{-1}K = {} & (\bar{A}_1^2 + \bar{B}_1^2)(\mu_1\lambda + \mu_2\sin 2\bar{\varphi}_1\lambda)\\ & + (\bar{A}_1\bar{A}_2 + \bar{B}_1\bar{B}_2)[\mu_3\sin(\bar{\varphi}_1 - \bar{\varphi}_2)\lambda + \mu_4\sin(\bar{\varphi}_1 + \bar{\varphi}_2)\lambda]\\ & + (\bar{A}_2^2 + \bar{B}_2^2)(\mu_5\lambda + \mu_6\sin 2\bar{\varphi}_2\lambda)\\ & + \mu_7\bar{\psi}_0\bar{A}_1\sin\bar{\varphi}_1\lambda + \mu_8\bar{\psi}_0\bar{A}_2\sin\bar{\varphi}_2\lambda + 2\bar{\psi}_0^2\lambda,\end{aligned}$$

$$\begin{aligned}(\tfrac{1}{2}h^2\bar{k}_1^2 c_{66})^{-1}U = {} & (\bar{A}_1^2 + \bar{B}_1^2)(\nu_1\lambda + \nu_2\sin 2\bar{\varphi}_1\lambda)\\ & + (\bar{A}_1\bar{A}_2 + \bar{B}_1\bar{B}_2)[\nu_3\sin(\bar{\varphi}_1 - \bar{\varphi}_2)\lambda + \nu_4\sin(\bar{\varphi}_1 + \bar{\varphi}_2)\lambda]\\ & + (\bar{A}_2^2 + \bar{B}_2^2)(\nu_5\lambda + \nu_6\sin 2\bar{\varphi}_2\lambda)\\ & + \nu_7\bar{\psi}_0\bar{A}_1\sin\bar{\varphi}_1\lambda + \nu_8\bar{\psi}_0\bar{A}_2\sin\bar{\varphi}_2\lambda + 2\bar{\psi}_0^2\lambda,\end{aligned}$$

where

$$\begin{aligned}
\mu_1 &= 1 + 3r\bar{\alpha}_1^2 & \nu_1 &= \bar{\beta}_1^2 + \bar{\gamma}_{11}\bar{\varphi}_1^2\\
\mu_2 &= (1 - 3r\bar{\alpha}_1^2)/2\bar{\varphi}_1 & \nu_2 &= (\bar{\beta}_1^2 - \bar{\gamma}_{11}\bar{\varphi}_1^2)/2\bar{\varphi}_1\\
\mu_3 &= 2(1 + 3r\bar{\alpha}_1\bar{\alpha}_2)/(\bar{\varphi}_1 - \bar{\varphi}_2) & \nu_3 &= 2(\bar{\beta}_1\bar{\beta}_2 + \bar{\gamma}_{11}\bar{\varphi}_1\bar{\varphi}_2)/(\bar{\varphi}_1 - \bar{\varphi}_2)\\
\mu_4 &= 2(1 - 3r\bar{\alpha}_1\bar{\alpha}_2)/(\bar{\varphi}_1 + \bar{\varphi}_2) & \nu_4 &= 2(\bar{\beta}_1\bar{\beta}_2 - \bar{\gamma}_{11}\bar{\varphi}_1\bar{\varphi}_2)/(\bar{\varphi}_1 + \bar{\varphi}_2)\\
\mu_5 &= 1 + 3r\bar{\alpha}_2^2 & \nu_5 &= \bar{\beta}_2^2 + \bar{\gamma}_{11}\bar{\varphi}_2^2\\
\mu_6 &= (1 - 3r\bar{\alpha}_1^2)/2\bar{\varphi}_2 & \nu_6 &= (\bar{\beta}_2^2 - \bar{\gamma}_{11}\bar{\varphi}_2^2)/2\bar{\varphi}_2\\
\mu_7 &= 4/\bar{\varphi}_1 & \nu_7 &= 4\bar{\beta}_1/\bar{\varphi}_1\\
\mu_8 &= 4/\bar{\varphi}_2 & \nu_8 &= 4\bar{\beta}_2/\bar{\varphi}_2.
\end{aligned}$$

The energy, per cycle, that escapes in the form of flexural waves in the unplated portions of the plate is equal to the work done, per cycle, by the transverse shear force and bending moment on sections of the unplated plate perpendicular to the x-axis:

$$E_l = -2\int_0^T \left(V_x\frac{\partial v}{\partial t} + M_x\frac{\partial \psi_x}{\partial t}\right) dt,$$

$$= 4\pi h^2 k_1^2 c_{66}(A_1^2 + B_1^2)(\hat{\gamma}_{11}\varphi_1 + \alpha_1\beta_1).$$

As the material is assumed to be lossless, E_l must be equal to the energy, per cycle, entering the plated portion of the plate:

$$E_a = \int_0^T\int_{-a}^{a} C\frac{\partial \bar{\psi}_x}{\partial t}\, dx\, dt,$$

$$= 4\pi h^2 \bar{k}_1^2 c_{66}(1 - \bar{\Omega}^2)\bar{\psi}_0(\bar{B}_1\bar{\varphi}_1^{-1}\sin\bar{\varphi}_1\lambda + \bar{B}_2\bar{\varphi}_2^{-1}\sin\bar{\varphi}_2\lambda).$$

A comparison of independent computations of E_l and E_a serves as a check on a major part of the entire computation.

The results of a series of computations of

$$Q = 2\pi(K + U)/E_l \tag{28}$$

for a partially plated AT-type plate for which $\omega_1/\bar{\omega}_1 = 1{\cdot}0354$ are illustrated in Fig. 3, where Q is plotted against a/h. As may be seen, Q is an almost periodic function of a/h with maxima exceeding 10^8 and minima ranging from about 10^3, for very short electrodes, to about 10^5 for a/h near Bechmann's Number which, as shown in the next section, is 14·3 for this case.

The maxima of Q occur, approximately, when the flexural mode in the plated portion has nodes at the boundaries; i.e. whenever an integral number of wave-lengths of the flexural mode just fits across the electrode:

$$a/h = m\pi/\bar{\Phi}_1, \qquad m = 1, 2, 3 \ldots,$$

where $\bar{\Phi}_1$ is given by (24) with $i = 1$ and the frequency that of resonance. The latter is given closely enough, for the present purpose, by $\bar{\Omega} = 1$. (The corresponding $\bar{\Phi}_1$ is illustrated in Fig. 1). Then, in terms of material properties and mass loading, the maxima of Q occur, approximately, when

$$a/h = m\pi/(\tilde{\gamma}_{11}^{-1} + 3r)^{\frac{1}{2}}, \qquad m = 1, 2, 3 \ldots$$

This is insensitive to R; so that, if we take $R = 0$, we find

$$a/h \approx m\pi/(3 + \pi^2 c_{66}/4\gamma_{11})^{\frac{1}{2}}, \; m = 1, 2, 3 \ldots \tag{29}$$

For the AT-cut of quartz, a/h, in (29), is about 1·6m. Thus, for the AT-cut, the maxima of Q occur at increments of electrode length of about 1·6 times the thickness of the plate.

Additional computations, with other mass ratios R, reveal that the locus of the minima of Q is raised as the mass per unit area of the electrodes is reduced.

In a physical plate, there would be additional losses which would affect the Q in a variety of ways. Losses through internal friction and the thermoelastic effect would probably have a quantitative, rather than a qualitative, influence—resulting in a great

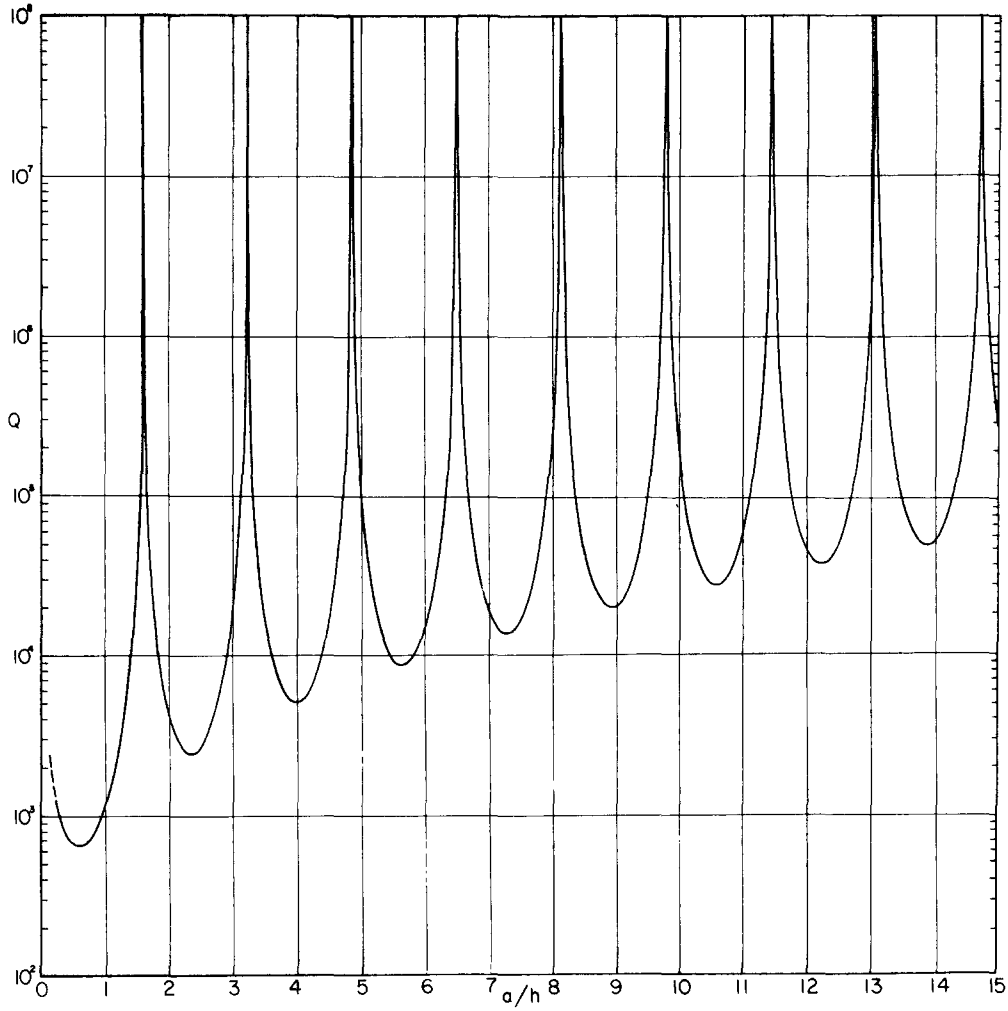

FIG. 3. Variation of Q as a result of variation of energy in escaping flexural waves as ratio of length of electrode to thickness of plate is changed.

reduction of the maxima of Q and a lesser reduction of the minima. On the other hand, losses through coupling of thickness-shear with modes other than flexure, neglected in the preceding analysis, would have a qualitative effect. Even in the case of straight-crested waves advancing in the digonal direction, there would be coupling of thickness-shear and flexure with face-shear. The energy in the escaping face-shear wave would have a periodicity different from that of the flexural wave— with a consequent destruction of the simple periodicity exhibited in Fig. 3. Moreover, the actual finite dimensions of the electrodes and plate would not permit simple, straight-crested waves. There would be some excitation of flexural, extensional and face-shear waves radiating laterally and extensional waves radiating longitudinally—each with its own periodicity of energy loss *versus* dimensional ratios. What might be expected in a series of experiments, then, would be an apparently erratic fluctuation of Q about a general diminution with reduction of areal dimensions (and increase of thickness) of electrodes.

4. BECHMANN'S NUMBER

In long, thin plates, the coupling of thickness-shear and flexural waves at a free boundary does not materially affect the frequencies of the fundamental thickness-shear mode and its anharmonic overtones in the neighborhood of the cut-off frequency [8]. Also, the flexural rigidity of the plate has little influence on the dispersion of thickness-shear waves at long wave-lengths. As a consequence of these two properties, the frequencies of the thickness-shear mode and its anharmonic overtones, in long, thin plates, are governed, to a good approximation, by equations of motion in which the term arising from the flexural rigidity of the plate is omitted—especially if, by the introduction of a suitable correction factor, the resulting dispersion relation is adjusted to nearly its original form: an adjustment which can be made very closely in a narrow frequency range [15].

The contribution of flexural rigidity, in (1) and (3), may be removed by omitting the first term in (1b) and in (3b). Then the equations of free vibration are

$$k_1^2 c_{66}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial \psi_x}{\partial x}\right) = -k_2\rho\omega^2 v, \tag{30a}$$

$$k_1^2 c_{66}\left(\frac{\partial v}{\partial x}+\psi_x\right) = \tfrac{1}{3}\rho\omega^2 h^2 \psi_x, \tag{30b}$$

for the unplated plate and, for the plated plate:

$$\bar{k}_1^2 c_{66}\left(\frac{\partial^2 \bar{v}}{\partial x^2}+\frac{\partial \bar{\psi}_x}{\partial x}\right) = -\bar{k}_2\rho\omega^2(1+R)\bar{v}, \tag{31a}$$

$$\bar{k}_1^2 c_{66}\left(\frac{\partial \bar{v}}{\partial x}+\bar{\psi}_x\right) = \tfrac{1}{3}\rho\omega^2 h^2(1+3R)\bar{\psi}_x, \tag{31b}$$

where k_2 and $\bar{k}_2$ are the factors introduced to correct the dispersion relations.

Equations (30) and (31) may be written in the forms

$$v = -\frac{h^2}{3k_2}\frac{\partial \psi_x}{\partial x}, \qquad \frac{h^2}{3k_2}\frac{\partial^2 \psi_x}{\partial x_2} - (1-\omega^2/\omega_1^2)\psi_x = 0, \tag{32}$$

$$\bar{v} = -\frac{h^2}{3r\bar{k}_2}\frac{\partial \bar{\psi}_x}{\partial x}, \qquad \frac{h^2}{3r\bar{k}_2}\frac{\partial^2 \bar{\psi}_x}{\partial x^2} + (\omega^2/\bar{\omega}_1^2-1)\bar{\psi}_x = 0. \tag{33}$$

To fix the correction factors k_2 and $\bar{k}_2$, we first find the dispersion relations from the approximate equations (32) and (33) by setting

$$\psi_x = A\exp(-\xi x), \qquad \bar{\psi}_x = \bar{A}\cos\bar{\xi}x,$$

with the results

$$\xi^2 h^2 = 3k_2(1-\omega^2/\omega_1^2), \qquad \bar{\xi}^2 h^2 = 3r\bar{k}_2(\omega^2/\bar{\omega}_1^2-1). \tag{34}$$

These give the correct frequencies of zero wave-number. As we are interested only in the frequency range $\bar{\omega}_1 \leqslant \omega \leqslant \omega_1$, we choose k_2 and $\bar{k}_2$ to give the correct wave-number ξ at $\omega = \bar{\omega}_1$ and the correct wave-number $\bar{\xi}$ at $\omega = \omega_1$:

$$k_2 = \Phi_2^2/3(1-\bar{\omega}_1^2/\omega_1^2), \qquad \bar{k}_2 = \bar{\Phi}_2^2/3r(\omega_1^2/\bar{\omega}_1^2-1), \tag{35}$$

where

$$\Phi_2 = [\varphi_2]_{\omega=\bar{\omega}_1}, \qquad \bar{\Phi}_2 = [\bar{\varphi}_2]_{\omega=\omega_1} \tag{36}$$

as calculated from (23) and (24), respectively, and identified in Fig. 1. Thus, k_2 and $\bar{k}_2$ are expressed in terms of material constants and mass loading. By this device, the dispersion relations for thickness-shear waves are returned, in the range $\bar{\omega}_1 \leqslant \omega \leqslant \omega_1$, to almost exactly their forms before omission of the flexural rigidities.

Turning, now, to the problem of vibrations of an infinite plate coated with electrodes over $-a \leqslant x \leqslant a$ (Fig. 2) we take

$$\psi_x|_{x>a} = A\exp[\xi(a-x)], \qquad \psi_x|_{x<-a} = A\exp[\xi(a+x)], \qquad \bar{\psi}_x = \bar{A}\cos\bar{\xi}x. \tag{37}$$

Upon substituting (37) into (32) and (33), we find

$$\omega^2 = \omega_1^2(1-\varphi^2/3k_2) = \bar{\omega}_1^2(1+\bar{\varphi}^2/3r\bar{k}_2), \tag{38}$$

where

$$\varphi = \xi h, \qquad \bar{\varphi} = \bar{\xi}h.$$

With the flexural rigidity omitted, the conditions of continuity at the boundaries of the plated and unplated portions of the plate reduce to

$$v = \bar{v}, \qquad V_x = \bar{V}_x \quad \text{on} \quad x = \pm a. \tag{39}$$

Upon substituting (37) into (39), and using (38), we find

$$\frac{A}{\bar{A}} = \frac{\bar{\varphi}k_2\sin\bar{\varphi}\lambda}{\varphi r\bar{k}_2} = \frac{\bar{k}_1^2\omega_1^2\cos\bar{\varphi}\lambda}{k_1^2\bar{\omega}_1^2}, \tag{40}$$

where, as before, $\lambda = a/h$. The second of (40) yields the frequency equation

$$\frac{a}{h} = \frac{1}{\bar{\varphi}}\tan^{-1}\left(\frac{r\bar{k}_1^2\bar{k}_2\omega_1^2\varphi}{k_1^2k_2\bar{\omega}_1^2\bar{\varphi}}\right). \tag{41}$$

The quantities $k_1, \bar{k}_1, k_2, \bar{k}_2, \omega_1, \bar{\omega}_1$ and r are determined solely by the material properties of the crystal and the mass ratio R; while φ and $\bar{\varphi}$ depend on the frequency through (38). Hence, for a given frequency, there corresponds a sequence of roots a/h equal to a constant multiplied by the integers $1, 2, 3, \ldots$; and only the constant changes for a different frequency. The computation of a frequency spectrum is thus a simple matter. An example, for $\omega_1/\bar{\omega}_1 = 1{\cdot}0354$, is illustrated in Fig. 4 along with experimental data obtained by Curran and Koneval [6]. (In the experiments, the width of the electrodes was small, in comparison with the length $2a$, instead of infinite as assumed in the mathematical solution. However, it was shown by Sykes, many years ago [16], that the thickness-shear frequencies are insensitive to width).

It may be seen, by inspection of (41) or Fig. 4, that the number of anharmonic overtones between $\bar{\omega}_1$ and ω_1 increases with increasing a/h. To find the values of a/h for which an additional overtone is included, it is only necessary to set $\omega = \omega_1$ in (41). Then,

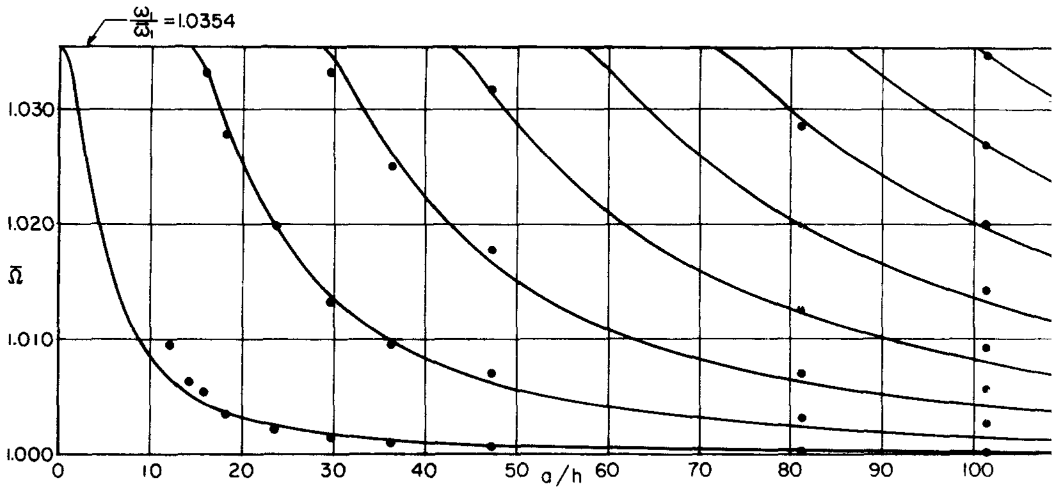

FIG. 4. Variation of number and distribution of resonances between cut-off frequencies. Comparison of theory with experiments by Curran and Koneval.

from (38), $\varphi = 0$ and, from (36), $\bar{\varphi} = \bar{\Phi}_2$; so that

$$a/h = m\pi/\bar{\Phi}_2, \qquad m = 1, 2, 3\ldots$$

where, from (24),

$$\bar{\Phi}_2^2 = \frac{\omega_1^2(1+3r\tilde{\gamma}_{11})}{2\bar{\omega}_1^2\tilde{\gamma}_{11}}\left[1-\left(1-\frac{12r\tilde{\gamma}_{11}(1-\bar{\omega}_1^2/\omega_1^2)}{(1+3r\tilde{\gamma}_{11})^2}\right)^{\frac{1}{2}}\right]. \tag{42}$$

The value of a/h, below which there are no anharmonic overtones, is, then,

$$a/h = \pi/\bar{\Phi}_2. \tag{43}$$

For the AT-type plate with $\omega_1/\bar{\omega}_1 = 1{\cdot}0354$, (43) yields $a/h = 14{\cdot}3$.

Equation (43) is to be compared with Curran and Koneval's

$$\frac{a}{h} = M\left(\frac{\bar{\omega}_1}{\omega_1-\bar{\omega}_1}\right)^{\frac{1}{2}}, \tag{44}$$

where M is a numerical constant estimated to be about 2·8. Equation (44) may be shown to be an asymptotic form of (43) for $R \ll 1$. In that case,

$$1-\bar{\omega}_1^2/\omega_1^2 \approx 2(1-\bar{\omega}_1/\omega_1) \ll 1, \qquad r \approx 1, \qquad \tilde{\gamma}_{11} \approx \hat{\gamma}_{11} = 4\gamma_{11}/\pi^2 c_{66}.$$

With these assumptions, (43) becomes

$$\frac{a}{h} \approx \pi\left(\frac{1}{6}+\frac{2\gamma_{11}}{\pi^2 c_{66}}\right)^{\frac{1}{2}}\left(\frac{\bar{\omega}_1}{\omega_1-\bar{\omega}_1}\right)^{\frac{1}{2}}. \tag{45}$$

Hence, in terms of the elastic properties of the plate, Curran and Koneval's constant M is

$$M = \pi\left(\frac{1}{6}+\frac{2\gamma_{11}}{\pi^2 c_{66}}\right)^{\frac{1}{2}}. \tag{46}$$

For the AT-cut of quartz, this formula gives $M = 2{\cdot}75$ and (45) gives $a/h = 14{\cdot}6$ for the plate with $\omega_1/\bar{\omega}_1 = 1{\cdot}0354$.

It may be remarked that the frequencies of the lowest branch in Fig. 4, although computed by the approximate method described in this section, match the frequencies computed in the preceding section extremely closely—even down to very small values of a/h. This is due to the fact that the deformation in the plated portion is almost entirely thickness-shear; i.e. very little of the reflection is in the form of flexure.

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Résumé—Dans cette étude, des solutions mathématiques sont présentées, et celles-ci prédisent l'influence de la longueur et de la masse de revêtements d'électrodes rectangulaires sur l'acuité de la résonance fondamentale "épaisseur-cisaillement" et sur le nombre et la distribution des vibrations anharmoniques entre les fréquences de détente des portions recouvertes et non recouvertes de plaques de quartz du type -AT.

Zusammenfassung—Mathematische Lösungen sind in dieser Abhandlung aufgezeigt welche den Einfluss der Länge und Masse von rechteckigen Elektrodenplatten auf die Schärfe der grundlegenden Scherungs Resonanz und auf die Anzahl und Verteilung von unharmonischen Obertönen zwischen den abgeschnittenen Frequenzen der gedeckfen und nicht-gedeckfen Teilen der AT-artigen Quarzplatten vorhersagen.

Абстракт—Настоящая работа демонстрирует математические решения, указывающие на влияние длины и массы прямоугольных гальванопокрытий на остроту основного резонанса поперечниа сдвигов и на количество и распределение ангармонических обертонов между граничными частотами покрытых и непокрытых частей кварцевых пластинок вида "AT".